Una nota sobre la transformada de Fourier en espacios de Hölder

A note on the Fourier transform in Hölder spaces

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Abstract. In this note we study the boundedness of the periodic Fourier transform from Lebesgue spaces into Hölder spaces. In particular, we generalize a classical result by Bernstein, [1]. **MSC 2010.** Primary 42A24, Secondary 42A16.

Resumen En este artículo, se estudia la acotación de la transformada periódica de Fourier desde espacios de Lebesgue a Espacios Hölder. Particularmente, se generaliza un resultado clásico de Bernstein.

Keywords: Hölder spaces, Fourier transform, Bernstein's theorem, Fourier series

Palabras Clave: espacios de Hölder, transformada de Fourier, espacios de Lebesgue

1. Introduction

Let us consider the periodic Fourier transform acting on measurable functions $f:\mathbb{T}\to\mathbb{C}$ by

$$(\mathscr{F}f)(n):=\widehat{f}(n)=\frac{1}{2\pi}\int_0^{2\pi}e^{-in\theta}f(\theta)d\theta,\ \ n\in\mathbb{Z},\tag{1}$$

where $\mathbb{T}=[0,2\pi)$ is the one-dimensional torus. As it is well known, if $f\in L^2(\mathbb{T})$, then $\mathscr{F}f\in L^2(\mathbb{Z})$ and $\|f\|_{L^2(\mathbb{T})}=\|\mathscr{F}f\|_{L^2(\mathbb{Z})}$. A generalization of this fact is the Hausdorff-Young inequality: if $f\in L^p(\mathbb{T})$, $1<p\leq 2$ then

$$\|\mathscr{F}f\|_{L^{p'}(\mathbb{Z})}\leq\|f\|_{L^p(\mathbb{T})},\ \ 1/p+1/p'=1.\tag{2}$$

Here, the periodic Hölder spaces are the Banach spaces defined for each $0<s\leq 1$ by

$$\Lambda^s(\mathbb{T})=\{f:\mathbb{T}\to\mathbb{C}:|f|_{\Lambda^s}=\sup_{x,h\in\mathbb{T}}|f(x+h)-f(x)||h|^{-s}<\infty\}$$

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together with the norm $\|f\|_{\Lambda^s} = |f|_{\Lambda^s} + \sup_{x\in\mathbb{T}} |f(x)|$. The main problem here is to determine which properties of f guarantees the p-summability of its periodic Fourier transform. In this topic, chronologically one should apparently start with the celebrated paper by S.N. Bernstein of 1914 [1] where he shows that if $f \in \Lambda^s(\mathbb{T})$, $1/2 < s < 1$, then $\mathscr{F}f \in L^1(\mathbb{Z})$. The Bernstein theorem is sharp: there exist functions in $\Lambda^{\frac{1}{2}}(\mathbb{T})$ whose Fourier transform does not converge absolutely. A classical example is the Hardy-Littlewood series

$$f(\theta) = \sum_{n=1}^{\infty} e^{in\log n} e^{in\theta}/n. \tag{3}$$

For $1 < p \leq \infty$ the inclusion map $i : L^1(\mathbb{Z}) \to L^p(\mathbb{Z})$ is continuous. Hence, by the Bernstein theorem, the Fourier transform is a bounded operator from $\Lambda^s(\mathbb{T})$ into $L^p(\mathbb{T})$ for all $1 < p \leq \infty$ and $\frac{1}{2} < s \leq 1$. Bernstein's theorem was generalized by O. Szász (see [9,10]) who proved that if $f \in \Lambda^{s,r}$ then $\mathscr{F}f \in L^p(\mathbb{Z})$, $p \geq 1$, where $s > 1/r + 1/p - 1$ if $1 < r \leq 2$ and $s > 1/p - 1/2$ if $r > 2$. Szász also gave examples to show that the range of values of s could not be extended. The Bernstein theorem and the Szász results has been extended to other groups. On the other hand, is known the Zygmund's result (see [5,19]) that the Hölder condition in Bernstein's theorem can be relaxed if f is of bounded variation. Zygmund shows that, in this case with f of bounded variation, and $f \in \Lambda^s(\mathbb{T})$, $0 < s < 1$, $\mathscr{F}f \in L^1(\mathbb{T})$. By the example of f as in (3), the boundedness of the Fourier transform fails from $\Lambda^s(\mathbb{T})$ into $L^1(\mathbb{Z})$. Other works regarding boundedness of the Fourier transform in Hölder spaces can be found in [11,12,13,14] and [15]. In this paper we obtain the following generalization of the Bernstein's theorem.

Theorem 1. *Let $2/3 < p \leq 2$ and let $s_p = 1/p - 1/2$. Then, the Fourier transform $f \mapsto \mathscr{F}f$ from $\Lambda^s(\mathbb{T})$ into $L^p(\mathbb{T})$ is a bounded operator for all s, $s_p < s < 1$. In particular, if $p = 1$ we obtain the Bernstein Theorem.*

We observe that the function $p \mapsto s_p$ from $(2/3, 2)$ into $(0, 1)$ is bijective; with this in mind one can combine Szász's results with our main theorem in order to give p-summability of the periodic Fourier transform on the interval $(2/3, \infty)$. However, we observe that for $p \geq 2$, the lower bound for s can be relaxed. We present this with more precision in the following remark.

Remark 1. Let $0 < s \leq \frac{1}{2}$. Then, the Fourier transform $\mathscr{F} : \Lambda^s(\mathbb{T}) \to L^p(\mathbb{T})$, is a bounded operator for $2 \leq p < \infty$, i.e, there exists a positive constant $C > 0$ satisfying $\|\mathscr{F}f\|_{L^p(\mathbb{Z})} \leq C\|f\|_{\Lambda^s(\mathbb{T})}$.

This note is organized as follows. In Section 2 we present some preliminaries and the corresponding statement of the Bernstein's theorem. In Section 3 we present the proof of our results, which we briefly discuss in the last section.

2. Preliminaries

In this section we introduce the necessary background in harmonic analysis used in the remainder of this paper. We first define the Fourier transform of certain discrete functions. Let $a \in L^2(\mathbb{T})$. The Fourier transform $\mathscr{F}(a) = \widehat{a}(\cdot)$ of a is the discrete function $\mathbb{Z}$ defined by.

$$\widehat{a}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} a(\theta) d\theta. \tag{4}$$

The Fourier inversion formula for Fourier series gives

$$a(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta}\, \widehat{a}(n). \tag{5}$$

The Plancherel formula for the Fourier periodic transformation gives

$$\sum_{m \in \mathbb{Z}} |\widehat{a}(m)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |a(\theta)|^2 d\theta. \tag{6}$$

Our main goal is the extension of a classical result proved by Bernstein on the periodic Fourier transform in Hölder functions. Thus, the corresponding statement is:

Theorem 2. (*Bernstein*). *If $f \in \Lambda^s(\mathbb{T})$, $\frac{1}{2} < s \leq 1$ then $\|\widehat{f}\|_{L^1(\mathbb{Z})} \leq C_s \|f\|_{\Lambda^s}$, i.e, the periodic Fourier transform extends to a bounded operator from $\Lambda^s(\mathbb{T})$ into $L^1(\mathbb{Z})$.*

Now, we are ready for the proof of our main results, i.e, Theorem 1 and Remark 1.

3. Proofs

Proof of Theorem 1. We begin by considering $t, h \in \mathbb{T}$, and $f \in \Lambda^s(\mathbb{T})$ for some $0 < s < 1$. Fourier inversion formulae guarantees that

$$f(t-h) - f(t) = \sum_{n \in \mathbb{Z}} (e^{-inh} - 1) \widehat{f}(n) e^{int}. \tag{7}$$

If take $h = 2\pi/3 \cdot 2^m$ and $2^m \leq n \leq 2^{m+1}$ we have

$$|e^{-inh} - 1| \geq \sqrt{3}. \tag{8}$$

By (8) and Plancherel formulae, we have

$$\begin{aligned} \sum_{2^m \leq n < 2^{m+1}} |\widehat{f}(n)|^2 &\leq \sum_{2^m \leq n < 2^{m+1}} |e^{-inh} - 1|^2 |\widehat{f}(n)|^2 \\ &= \|f(\cdot - h) - f(\cdot)\|^2_{L^2(\mathbb{T})} \\ &\leq \|f(\cdot - h) - f(\cdot)\|^2_{L^\infty(\mathbb{T})} \\ &\leq \left[\frac{2\pi}{3 \cdot 2^m}\right]^{2s} |f|^2_{\Lambda^s(\mathbb{T})}. \end{aligned}$$

Now we consider the case of the boundedness of $\mathscr{F}$ from Λ^s into L^p for $2/3 < p < 2$, and $s_p < s < 1$. Let $\varepsilon > 0$ be such that $p = 2 - \varepsilon$. In this case, $0 < \varepsilon < \frac{4}{3}$ and $s_p = \frac{1}{2}\varepsilon(2-\varepsilon)^{-1} < s < 1$. If $r = 2(2-\varepsilon)^{-1}$ then $r > 1$. By Hölder inequality we have,

$$\begin{aligned}\sum_{2^m \leq n < 2^{m+1}} |\widehat{f}(n)|^{2-\varepsilon} &\leq \left(\sum_{2^m \leq n < 2^{m+1}} |\widehat{f}(n)|^{(2-\varepsilon)r}\right)^{\frac{1}{r}} \cdot \left(\sum_{2^m \leq n < 2^{m+1}} 1\right)^{\frac{1}{r'}} \\ &\leq \left(\sum_{2^m \leq n < 2^{m+1}} |\widehat{f}(n)|^{2}\right)^{\frac{1}{r}} \cdot 2^{(m+1)/r'} \\ &\leq \left[2\pi/3 \cdot 2^m\right]^{2s/r} \cdot 2^{(m+1)/r'} |f|^{2/r}_{\Lambda^s(\mathbb{T})} \\ &= \left[2\pi/3\right]^{2s/r} 2^{(m+1)/r' - 2ms/r} |f|^{2/r}_{\Lambda^s(\mathbb{T})}.\end{aligned}$$

First note that

$$\frac{m+1}{r'} - \frac{2ms}{r} = m\left(\frac{\varepsilon}{2} + s\varepsilon - 2s\right) + \frac{\varepsilon}{2}. \tag{9}$$

From the conditions $0 < \varepsilon < 4/3$ and $\frac{\varepsilon}{2}(2-\varepsilon)^{-1} < s < 1$ we obtain the relation: $\frac{\varepsilon}{2} + s\varepsilon - 2s < 0$. Hence,

$$\begin{aligned}\sum_{n \geq 1} |\widehat{f}(n)|^{2-\varepsilon} &= \sum_{m=0}^{\infty} \sum_{2^m \leq n < 2^{m+1}} |\widehat{f}(n)|^{2-\varepsilon} \\ &\leq \sum_{m=0}^{\infty} 2^{m(\frac{\varepsilon}{2} + s\varepsilon - 2s)} \cdot 2^{\varepsilon/2} \left[2\pi/3\right]^{2s/r} |f|^{2/r}_{\Lambda^s(\mathbb{T})} \\ &\leq C |f|^{2/r}_{\Lambda^s(\mathbb{T})}.\end{aligned}$$

For $n \leq -1$ we recall the formula $\overline{\widehat{f}(-n)} = \widehat{\overline{f}}(n)$. So, we get,

$$\begin{aligned}\sum_{n \leq -1} |\widehat{f}(n)|^{2-\varepsilon} = \sum_{n \geq 1} |\widehat{f}(-n)|^{2-\varepsilon} &\leq C |\overline{f}|^{2/r}_{\Lambda^s(\mathbb{T})} = C |f|^{2/r}_{\Lambda^s(\mathbb{T})} \\ &\leq C \|f\|^{2/r}_{\Lambda^s(\mathbb{T})}.\end{aligned}$$

Remembering that $|\widehat{f}(0)| \leq \|f\|_{\Lambda^s(\mathbb{T})}$ we can write

$$\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^{2-\varepsilon} \leq C' \|f\|^{2/r}_{\Lambda^s(\mathbb{T})}. \tag{10}$$

Hence,

$$\begin{aligned}\|\mathscr{F} f\|_{L^p(\mathbb{Z})} &= \left(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^{2-\varepsilon}\right)^{(2-\varepsilon)^{-1}} \\ &\leq C^{(2-\varepsilon)^{-1}} \|f\|^{\frac{2}{r}(2-\varepsilon)^{-1}}_{\Lambda^s(\mathbb{T})} = C^{(2-\varepsilon)^{-1}} \|f\|_{\Lambda^s(\mathbb{T})}.\end{aligned}$$

Finally, we consider the boundedness of $\mathscr{F}$ when $p = 2$. In this case for $0 < s < 1$, by the Plancherel formula, we get:

$$\|\mathscr{F} f\|_{L^2(\mathbb{Z})} \leq \|f\|_{L^2(\mathbb{T})} \leq C\|f\|_{\Lambda^s(\mathbb{T})}.$$

Proof of Remark 1. Let us consider $1 < q \leq 2$ the corresponding conjugated exponent of p, i.e, the unique real number satisfying $1/p + 1/q = 1$. By the Hausdorff-Young inequality we have $\|\widehat{f}\|_{L^p(\mathbb{T})} \leq \|f\|_{L^q(\mathbb{T})}$. Moreover,

$$\begin{aligned}
&\|\widehat{f}\|_{L^p(\mathbb{T})} \leq \|f\|_{L^q(\mathbb{T})} \\
&\quad = (\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^q)^{1/q} dx \\
&\quad \leq (\frac{1}{2\pi}\int_0^{2\pi} |f(x) - f(0)|^q dx)^{1/q} + \frac{1}{2\pi}|f(0)| \\
&\quad = (\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(x) - f(0)|^q}{|x-0|^{qs}} |x|^{qs} dx)^{1/q} + \frac{1}{2\pi}|f(0)| \\
&\quad \leq \sup_{x\in\mathbb{T}} \frac{|f(x) - f(0)|}{|x-0|^s} (\int_0^{2\pi} x^{sq} dx)^{1/q} + \frac{1}{2\pi}\sup_{x\in\mathbb{T}}|f(x)| \\
&\quad \leq C\|f\|_{\Lambda^s(\mathbb{T})}.
\end{aligned}$$

4. Discussion

In this paper we generalize a classical theorem, published in 1914 by Bernstein. The Bernstein's theorem gives a sufficient condition for the summability of the periodic Fourier transform of functions on the circle, by imposing certain regularity conditions on such functions. More precisely Bernstein's theorem guarantees that regularity of order $s \in (1/2, 1]$ is sufficient. Our Theorem 1 gives the $p-$summability of the Fourier transform, $p \in (2/3, 1]$ by imposing regularity of order $s \in (s_p, 1]$ where $s_p = 1/p - 1/2$. Particularly if $p = 1$ we obtain the Bernstein's theorem. Additionally, we note in Remark 1 that for the $p-$ summability of the Fourier transformation, $p \geq 2$, we need $s \in (0, 1)$. It is possible extend this topic to the case of general compact Lie groups by using representation theory. This would be part of a future work. Recent works regarding the summability of the Fourier transform can be found in [3,4,6,7,17,18].

References

1. Bernstein, S. Sur la convergence absolue des séries trigonométriques. Comptes Rendum Hebdomadaires des Séances de l'Academie des Sciences, Paris, 158, 1661-1663, (1914).
2. Bloom, W. R. Bernstein's inequality for locally compact Abelian groups. Journal the Australian mathematical society 17, 88-101, (1974).

3. Gát, G., Nagy, K. On the logarithmic summability of Fourier series. Georgian Math. J. 18 (2011)
4. Getsadze, R. On Cesáro summability of Fourier series with respect to double complete orthonormal systems. J. Anal. Math. 102, 209-223. (2007).
5. Katznelson, Y.: An introduction to Harmonic Analysis. Cambridge University Press (2004)
6. Khasanov, Y. K. On the absolute summability of Fourier series of almost periodic functions. Translation of Ukraïn. Mat. Zh. 65 (2013), no. 12, 1716-1722. Ukrainian Math. J. 65 (12), 1904-1911. (2014).
7. Khasanov, Y. K. On absolute summability of Fourier series of almost periodic functions. (Russian) Anal. Math. 39, no. 4, 259-270. (2013)
8. Onnewer, C. W. Absolute convergence of Fourier series on certain groups. Duke Mathematical Journal. 39, 599-609, (1972)
9. Szász, O. Über den Konvergenzexponenten der Fouriersohen Reihen gewisser Funktionenklassen, Sitzungsberichte der Bayerischen Akademie der Wissenschaften Mathematisch-physikalische Klasse. 135-150, (1922).
10. Szász, O. Über die Fourierschen gewiser Funktionenklassen. Mathematische Annalen. 530-536. (1928)
11. Szász, Otto Zur Konvergenztheorie der Fourierschen Reihen. (German) Acta Math. 61 (1933), no. 1, 185-201.
12. Szász, O. Fourier series and mean moduli of continuity, Trans. American Math. Soc., 42, 366-395 (1937)
13. Titchmarsh, E. C. A note on the Fourier transform, Journal of the London Mathematical Society. 2, 148-150. (1963).
14. Ogata, N. On the absolute convergence of Lacunary Fourier Series. Scientiae Mathematicae. 2(3), 337-343 (1949).
15. Leinder, L. Comments on the absolute convergence of Fourier series. Hokkaido Mathematical Journal. 30, 221-230 (2001)
16. Li, L. Zhang, Y. The Cesáro summability of Fourier series on Hardy spaces. J. Math. (Wuhan) 27 (2007), no. 1, 1-9.
17. Weisz, F. Restricted summability of Fourier series and Hardy spaces. Acta Sci. Math. (Szeged) 75, no. 1-2, 197-217. (2009)
18. Young, W. H. On Classes of Summable Functions and their Fourier Series. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 87(594), 225-229, (1912).
19. Zygmund, A.: *Trigonometric series.* 2nd ed., Cambridge University Press (1959)